

# THE NAMBU-GOLDSTONE MODES OF BOSE-EINSTEIN CONDENSED TWO-DIMENSIONAL MAGNETOEXCITONS

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## Abstract

The collective elementary excitations of the two-dimensional (2D) electron-hole systems in a strong perpendicular magnetic field are discussed from the point of view of the Bogoliubov [1] and Goldstone [2] theorems concerning the many-body Hamiltonian with continuous symmetries, continuously degenerate ground states, forming a ring of minima on the energy scale in dependence on the phase of the field operator. This system due to the quantum fluctuations does select a concrete ground state with a fixed phase of the field operator forming a ground state with a spontaneously broken continuous symmetry [1-3]. The collective excitation of this new ground state related only with the changes of the field operator phase without changing its amplitude leads to the quantum transitions along the ring of the minima and does not need excitation energy in the long wavelength limit. This type of gapless excitations is referred to as Nambu-Goldstone modes [2-8]. They are equivalent to massless particles in the relativistic physics. The concrete realization of these theorems in the case of 2D magnetoexcitons with direct implications of the plasmon-type excitations side by side with the exciton ones is discussed below in terms of the Bogoliubov [1] theory of quasiaverages.

## 1. Introduction: The Bogoliubov's Theory of the Quasiaverages

Bogoliubov [1] demonstrated his concept of quasiaverages by using the ideal Bose-gas model with the Hamiltonian

$$H = \sum_k \left( \frac{\hbar^2 k^2}{2m} - \mu \right) a_k^\dagger a_k, \quad (1)$$

here  $a_k^\dagger$ ,  $a_k$  are the Bose operators of creation and annihilation of particles, and  $\mu$  is their chemical potential.

The occupation numbers of the particles are

$$N_0 = \frac{1}{e^{-\beta\mu} - 1}; \quad N_k = \frac{1}{e^{\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)} - 1}, \quad (2)$$

where  $\mu \leq 0$  and  $\beta = 1/kT$ .

In the normal state, the density of particles in the thermodynamic limit at  $\mu = 0$  becomes  $n = 2.612(mk_B T)^{3/2} / (2\pi\hbar^2)^{3/2}$ . At this point, the Bose-Einstein condensation occurs and a finite value of the density of condensed particles appears in the thermodynamic limit

$$n_0 = \lim_{V \rightarrow \infty} \frac{N_0}{V}; \quad \mu = -k_B T \ln \left( 1 + \frac{1}{N_0} \right) \quad (3)$$

The operators  $a_0^\dagger$  and  $a_0$  asymptotically become  $c$ -numbers, when their commutator

$$\left[ \frac{a_0}{\sqrt{V}}, \frac{a_0^\dagger}{\sqrt{V}} \right] = \frac{1}{V} \quad (4)$$

asymptotically tends to zero and their product is equal to  $n_0$ . One can then write

$$\frac{a_0^\dagger}{\sqrt{V}} \square \sqrt{n_0} e^{i\alpha}; \quad \frac{a_0}{\sqrt{V}} \square \sqrt{n_0} e^{-i\alpha} \quad (5)$$

On the other hand, the regular averages of the operators  $a_0^\dagger$  and  $a_0$  in Hamiltonian (1) are exactly equal to zero. It is the consequence of the commutativity of the operator  $H$  and the operator of the total particle number  $N$  as follows

$$\hat{N} = \sum_k a_k^\dagger a_k; \quad [H, \hat{N}] = 0. \quad (6)$$

As a result, the operators  $H$  is invariant with respect to the unitary transformation

$$U = e^{i\hat{N}\phi} \quad (7)$$

with an arbitrary angle  $\phi$ . This invariance is called gradient invariance of the first kind or gauge invariance. When  $\phi$  does not depend on the coordinate  $x$ , we have the global gauge invariance and in the case  $\phi(x)$  it is named as local gauge invariance [2-8] or gauge invariance of the second kind.

The invariance (7) implies  $H = U^\dagger H U$ ;  $U^\dagger a_0 U = e^{i\phi} a_0$ , which leads to the following average value

$$\langle a_0 \rangle \cong \text{Tr}(a_0 e^{-\beta H}) = \text{Tr}(a_0 U e^{-\beta H} U^\dagger) = \text{Tr}(U^\dagger a_0 U e^{-\beta H}) = e^{i\phi} \langle a_0 \rangle;$$

$$(1 - e^{i\phi}) \langle a_0 \rangle = 0$$

The selection rules arise

$$\langle a_0 \rangle = 0; \quad \langle a_0^\dagger \rangle = 0 \quad (8)$$

because  $\phi$  is an arbitrary angle. The regular average (8) can also be obtained from asymptotical expressions (5) if they are integrated over the angle  $\alpha$ . This apparent contradiction can be resolved if Hamiltonian (1) is completed by an additional term

$$-v(a_0^\dagger e^{i\phi} + a_0 e^{-i\phi})\sqrt{V}, \quad v > 0, \quad (9)$$

where  $\phi$  is a fixed angle and  $v$  an infinitesimal value.

The new Hamiltonian has the form

$$H_{v,\phi} = \sum_k \left( \frac{\hbar^2 k^2}{2m} - \mu \right) a_k^\dagger a_k - v(a_0^\dagger e^{i\phi} + a_0 e^{-i\phi})\sqrt{V}. \quad (10)$$

It does not conserve the condensate number. Now the regular average values of the

operators  $a_0^\dagger$  and  $a_0$  over the Hamiltonian  $H_{\nu,\phi}$  differ from zero, i.e.,  $\langle a_0 \rangle_{H\varphi} \neq 0$  and  $\langle a_0^\dagger \rangle_{H\varphi} \neq 0$ . The definition of the quasiaverages designated by  $\langle a_0 \rangle$  is the limit of the regular average  $\langle a_0 \rangle_{H\varphi}$  when  $\nu$  tends to zero

$$\langle a_0 \rangle = \lim_{\nu \rightarrow 0} \langle a_0 \rangle_{H_{\nu,\varphi}}. \quad (11)$$

It is important to emphasize that the limit  $\nu \rightarrow 0$  must be effectuated after the thermodynamic limit  $V \rightarrow \infty$ ,  $N_0 \rightarrow \infty$ . In the thermodynamic limit,  $\mu$  is also infinitesimal, and it is possible to choose the ratio of two infinitesimal values  $\mu$  and  $\nu$  to give a finite value

$$-\frac{\nu}{\mu} = \sqrt{n_0}. \quad (12)$$

To calculate the regular average  $\langle a_0 \rangle_{H_{\nu,\varphi}}$  one needs to represent the Hamiltonian (10)  $H_{\nu,\phi}$  in a diagonal form with the aid of the canonical transformation over the amplitudes

$$a_0 = -\frac{\nu}{\mu} e^{i\varphi} \sqrt{V} + \alpha_0; \quad a_k = \alpha_k; \quad k \neq 0. \quad (13)$$

In terms of the new variables the Hamiltonian  $H_{\nu,\phi}$  takes the form

$$H_{\nu,\phi} = -\mu \alpha_0^\dagger \alpha_0 + \sum_k \left( \frac{\hbar^2 k^2}{2m} - \mu \right) \alpha_k^\dagger \alpha_k + \frac{\nu^2 V}{\mu}. \quad (14)$$

In the diagonal representation (14), the regular average value  $\langle \alpha_0 \rangle_{H_{\nu,\varphi}}$  exactly equals zero, while the value  $\langle a_0 \rangle_{H_{\nu,\varphi}}$  equals the first term on the right-hand side of formulas (13).

As a result, the quasiaverage  $\langle a_0 \rangle$  is

$$\langle a_0 \rangle = \lim_{\nu \rightarrow 0} \langle a_0 \rangle_{H_{\nu,\varphi}} = \sqrt{N_0} e^{i\varphi}. \quad (15)$$

It depends on the fixed angle  $\varphi$  and does not depend on  $\nu$ . The spontaneous global gauge symmetry breaking was implied when the phase  $\varphi$  of the condensate amplitude in Hamiltonian (10) was fixed.

When the interaction between the particles is taken into account, these differences appear for other amplitudes as well. They give rise to the renormalization of the energy spectrum of the collective elementary excitations. In such a way, the canonical transformation

$$a_k = \sqrt{N_0} \delta_{k,0} e^{i\varphi} + \alpha_k \quad (16)$$

introduced for the first time by Bogoliubov [1] in his theory of superfluidity, has a quantum-statistical foundation within the framework of the quasiaverage concept. At  $T=0$  the quasiaverage  $\langle a_0 \rangle$  coincides with the average over the quantum-mechanical ground state, which is a coherent macroscopic state [9].

The phenomena related with the spontaneous breaking of the continuous symmetry play an important role in statistical physics. The first contributions in this direction belonging to Nambu [3], Goldstone [2], Higgs [4], and Weinberg [5] were stimulated under the influence of the theory of superconductivity originated by Bardeen, Cooper, and Schrieffer [6] and refined by Bogoliubov [1]. Some elements of these concepts, such as the coherent macroscopic state with a given fixed phase and the displacement canonical transformation of the field operator describing the Bose-Einstein condensate, were introduced by Bogoliubov in the microscopical theory of

superfluidity [1] and were generalized in his theory of quasiaverages [1] noted above.

The brief review of the gauge symmetries, their spontaneous breaking, Goldstone and Higgs effects will be presented below following the Ryder's monograph [7] and Berestetskii's lectures [8].

## 2. The Goldstone's Theorem

Goldstone demonstrated his main idea considering a simple model of a complex scalar Bose field, which in classical description has the Lagrangian

$$L = \left( \frac{\partial \phi^*}{\partial x_\mu} \right) \left( \frac{\partial \phi}{\partial x^\mu} \right) - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2. \quad (17)$$

The potential energy  $V(\phi)$  has the form

$$V(\phi) = m^2 \phi^* \phi + \lambda (\phi^* \phi)^2; \quad \lambda > 0, \quad (18)$$

where  $m^2$  is regarded as a parameter only, rather than a mass term,  $\lambda$  is the parameter of self-interaction, whereas the denotations  $x_\mu$  and  $x^\mu$  mean

$$x^\mu = (ct, \vec{x}); \quad x_\mu = (ct, -\vec{x}); \quad (19)$$

Lagrangian is invariant under the global gauge transformation

$$\phi = e^{i\Lambda} \phi'; \quad L(\phi) = L(\phi'); \quad \Lambda - \text{constant}. \quad (20)$$

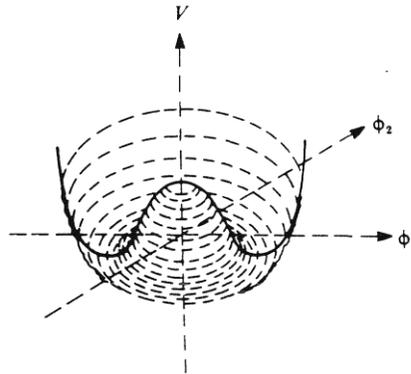
It has a global gauge symmetry. The ground state is obtained by minimizing the potential as follows

$$\frac{\partial V(\phi)}{\partial \phi} = m^2 \phi^* + 2\lambda \phi^* |\phi|^2. \quad (21)$$

The interest presents the case  $m^2 < 0$ , when the minima are situated along the ring

$$|\phi|^2 = -\frac{m^2}{2\lambda} = a^2; \quad |\phi| = a; \quad a > 0. \quad (22)$$

The function  $V(\phi)$  is shown in Fig. 1 being plotted against two real components of the field  $\phi_1$  and  $\phi_2$ .



**Fig. 1.** The potential  $V(\phi)$  with the minima at  $|\phi| = a$  and a local maximum at  $\phi = 0$ .

There is a set of degenerate vacua related to each other by rotation. The complex scalar field can be expressed in terms of two scalar real fields, such as  $\rho(x)$  and  $\theta(x)$ , in polar

coordinates representation or in Cartesian decomposition as follows

$$\phi(x) = \rho(x)e^{i\theta(x)} = (\phi_1(x) + i\phi_2(x)) \frac{1}{\sqrt{2}}. \quad (23)$$

The Bogoliubov-type canonical transformation breaking the global gauge symmetry was written as

$$\phi(x) = a + \frac{\phi'_1(x) + i\phi'_2(x)}{\sqrt{2}} = (\rho'(x) + a)e^{i\theta'(x)}. \quad (24)$$

The new particular vacuum state has the average  $\langle \phi \rangle_0 = a$  with the particular vanishing vacuum expectation values  $\langle \phi'_1 \rangle_0 = \langle \phi'_2 \rangle_0 = \langle \rho' \rangle_0 = \langle \theta' \rangle_0 = 0$ . It means the selection of one vacuum state with infinitesimal phase  $\theta' \rightarrow 0$ . As was mentioned in [7], the physical fields are the excitations above the vacuum. They can be realized by performing perturbations about  $|\phi| = a$ . Expanding Lagrangian (17) in series on the infinitesimal perturbations  $\theta'$ ,  $\rho'$ ,  $\phi'_1$ ,  $\phi'_2$  ignoring the constant terms, we will obtain

$$L = \frac{1}{2}(\partial_\mu \phi'_1)(\partial^\mu \phi'_1) + \frac{1}{2}(\partial_\mu \phi'_2)(\partial^\mu \phi'_2) - 2\lambda a^2 \phi_1'^2 - \sqrt{2}\lambda \phi'_1(\phi_1'^2 + \phi_2'^2) - \frac{\lambda}{4}(\phi_1'^2 + \phi_2'^2)^2 \quad (25)$$

or in polar description

$$L = (\partial_\mu \rho')(\partial^\mu \rho') + (\rho' + a)^2 (\partial_\mu \theta')(\partial^\mu \theta') - [\lambda \rho'^4 + 4a\lambda \rho'^3 + 4\lambda a^2 \rho'^2 - \lambda a^4] \quad (26)$$

Neglecting the cubic and quartic terms, we will see that there are the quadratic terms only of the type  $4\lambda a^2 \rho'^2$  and  $2\lambda a^2 \phi_1'^2$ , but there are not quadratic terms proportional to  $\theta'^2$  and  $\phi_2'^2$ . If we compare these Lagrangians with relativistic physics, we can conclude that the field components  $\phi'_1$  and  $\rho'$  represent massive particles and dispersion laws with energy gap, whereas the field components  $\phi'_2$  and  $\theta'$  represent the massless particles and gapless energy spectrum.

The main Goldstone results can be formulated as follows

$$\begin{aligned} m_{\rho'}^2 &= 4\lambda a^2, & m_{\phi_1'}^2 &= 2\lambda a^2; \\ m_{\phi_2'}^2 &= 0; & m_{\theta'}^2 &= 0; \end{aligned} \quad (27)$$

The spontaneous breaking of the global gauge symmetry takes place due to the influence of the quantum fluctuations. They transform the initial field  $\phi$  with two massive real components  $\phi_1$  and  $\phi_2$ , and a degenerate ground state with the minima forming a ring into another field with one massive and other massless components, the ground state of which has a well defined phase without initial symmetry.

The elementary excitations above the new ground state changing the value  $\langle \rho \rangle = a$  are massive. It costs the energy to displace  $\rho'$  against the restoring forces of the potential  $V(\rho)$ . But there are no restoring forces corresponding to displacements along the circular valley  $|\phi| = a$  formed by initial degenerate vacua.

Hence, for angular excitations  $\theta'$  of wavelength,  $\lambda$  we have  $\omega \propto \lambda^{-1} \rightarrow 0$  as  $\lambda \rightarrow \infty$ . The dispersion law is  $\omega \propto ck$  and the particles are massless [7]. The  $\theta'$  particles are known as the Goldstone bosons. This phenomenon is general and takes place in any order of perturbation theory. The spontaneous breaking of a continuous symmetry not only of the type as a global

gauge symmetry but also of the type of rotational symmetry entails the existence of massless particles referred to as Goldstone particles or Nambu-Goldstone gapless modes. These statements are known as Goldstone theorem. Its affirms that there exists a gapless excitation mode when a continuous symmetry is spontaneously broken. The angular excitations  $\theta'$  are analogous to the spin waves. The latter represent a slow spatial variation of the direction of magnetization without changing of its absolute value. Since the forces in a ferromagnetic are of short range, it requires a very little energy to excite this ground state. So, the frequency of the spin waves has the dispersion law  $\omega = ck$ . As was mentioned by Ryder [7], this argument breaks down if there are long-range forces like, for example, the  $1/r$  Coulomb force. In this case, we deal with the maxwellian gauge field with local depending on  $x$  gauge symmetry instead of global gauge symmetry considered above.

The case of Goldstone field  $\phi$  and of a maxwellian field with local gauge symmetry will be discussed below. But before it, a specific application of the above statement will be demonstrated following References [10-16], where the spinor Bose-Einstein condensates are discussed.

### 3. The Bogoliubov Excitations and the Nambu-Goldstone Modes

The above formulated theorems can be illustrated using the specific example of the Bose-Einstein condensed sodium atoms  $^{23}\text{Na}$  in an optical-dipole trap following the investigations of Murata, Saito and Ueda [10] on the one side and of Uchino, Kobayashi and Ueda [11] on the other side. There are many other references in this direction, among which, we can mention [12-17]. The sodium atoms  $^{23}\text{Na}$  have a spin  $f = 1$  of the hyperfine interaction and obey the Bose statistics. The interacting bosons with  $f = 1$  have a resultant spin  $F$  with the values  $F = 0, 1, 2$ . The contact hard-core interaction constant  $g_F = 4\pi\hbar^2 a_F / M$  are characterized by s-wave scattering lengths  $a_F$ , which are different from zero for  $F = 0$  when two atomic spins form a singlet, and for  $F = 2$ , when they form a quintuplet. The constant  $g_0$  and  $g_2$  enter into the combinations  $c_0 = (g_0 + 2g_2)/3$  and  $c_1 = (g_2 - g_0)/3$  which determine the Hamiltonian. The description of the atomic Bose gas in an optical-dipole trap is possibly in the plane-wave representation due to the homogeneity and the translational symmetry of the system. It means that the components of the Bose field operator  $\psi_m(\vec{r})$  can be represented in the form:

$$\psi_m(\vec{r}) = \frac{1}{\sqrt{V}} \sum_k a_{km} e^{i\vec{k}\vec{r}} \quad (28)$$

where  $a_{km}$  is the annihilation operator with a wave vector  $\vec{k}$  and a magnetic quantum number  $m$ , which in the case  $f = 1$  takes three values 1, 0, -1.  $V$  is the volume of the system. The spinor Bose-Einstein condensates were realized experimentally by the MIT group [12] in different spin combinations using the sodium atoms  $^{23}\text{Na}$  in a hyperfine spin states  $|f = 1, m_f = -1\rangle$  in a magnetic trap and then transforming them to an optical-dipole trap formed by a single infrared laser. The Bose-Einstein condensates were found to be long-lived. Some arguments concerning the metastable long-lived states were formulated. They may appear if the energy barriers, which prevent the system from a direct evolving toward its ground states, do exist. If the thermal energy needed to overcome these barriers is not available, the metastable state may be long-lived and these events are commonly encountered. Even the Bose-Einstein condensates in the dilute atomic

gases also arise due to the metastability. More so, in the gases with attractive interactions the Bose-Einstein condensates may be metastable against the collapse just due to the energy barriers [12]. Bellow we will discuss the Bogoliubov-type collective elementary excitations arising over the metastable long-lived ground states of the spinor-type Bose-Einstein condensates (BEC-tes) following [10, 11], so as to demonstrate the formation of the Nambu-Goldstone modes.

The Hamiltonian considered in [10] is given by formulas (3) and (4), and has the form

$$H = \sum_{\vec{k}, m} (\varepsilon_{\vec{k}} - pm + qm^2) a_{\vec{k}m}^\dagger a_{\vec{k}m} + \frac{c_0}{2V} \sum_{\vec{k}} : \hat{\rho}_{\vec{k}}^\dagger \hat{\rho}_{\vec{k}} : + \frac{c_1}{2V} \sum_{\vec{k}} : \hat{f}_{\vec{k}}^\dagger \hat{f}_{\vec{k}} : \quad (29)$$

Here the denotations were used

$$\begin{aligned} \varepsilon_k &= \frac{\hbar^2 k^2}{2M}; \quad c_0 = (g_0 + 2g_2)/3; \quad c_1 = (g_2 - g_0)/3 \\ \hat{\rho}_{\vec{k}} &= \sum_{\vec{q}, m} a_{\vec{q}, m}^\dagger a_{\vec{q}+\vec{k}, m}; \quad \hat{f} = (\hat{f}^x, \hat{f}^y, \hat{f}^z); \\ \hat{f}_{\vec{k}} &= \sum_{q, m, n} \hat{f}_{mn} a_{q, m}^\dagger a_{q+k, m} \\ \hat{f}^x &= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \frac{1}{\sqrt{2}}; \quad \hat{f}^y = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} \frac{i}{\sqrt{2}}; \quad \hat{f}^z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} \end{aligned} \quad (30)$$

The repeated indices are assumed to be summed over 1,0,-1. The symbol  $::$  denotes the normal ordering of the operators. The coefficient  $p$  is the sum of the linear Zeeman energy and of the Lagrangian multiplier, which is introduced to set the total magnetization in the  $z$  direction to a prescribed value. This magnetization is conserved due to the axisymmetry of the system in a magnetic field.  $q$  is the quadratic Zeeman effect energy, which is positive in the case of spin  $f=1$   $^{23}\text{Na}$  and  $^{87}\text{Rb}$  atoms. The spin-spin interaction is ferromagnetic-type with  $c_1 < 0$  for the  $f=1$   $^{87}\text{Rb}$  atoms and is antiferromagnetic-type with  $c_1 > 0$  for the  $f=1$   $^{23}\text{Na}$  atoms [10]. Taking into account the fact that in many experimental situations the linear Zeeman effect can be ignored and the quadratic Zeeman effect term  $q$  can be manipulated experimentally, in [11] the both cases of positive and negative  $q$  at  $p=0$  were investigated for the spin  $-1$  and spin  $-2$  Bose-Einstein condensates (BECs). We will confine ourselves to the review of some spinor phases with spin  $-1$  discussed in [11] so as to demonstrate the relations between the Nambu-Goldstone(NG) modes of the Bogoliubov energy spectra and the spontaneous breaking of the continuous symmetries. The description of the excitations is made in [10, 11] in the number-conserving variant of the Bogoliubov theory [1]. There is no need to introduce the chemical potential as a Lagrangian multiplier in order to adjust the particle number to a prescribed value.

The BEC takes place on a superposition state involving the single-particle states with wave vector  $\vec{k} = 0$  and different magnetic quantum numbers

$$|\xi\rangle = \sum_m \xi_m a_{0, m}^\dagger |vac\rangle; \quad \sum_m |\xi_m|^2 = 1 \quad (31)$$

The order parameter has a vector form and consists of three components:  $\vec{\xi} = (\xi_1, \xi_0, \xi_{-1})$ . The vacuum state  $|vac\rangle$  means the absence of the atoms. The ground state wave function of the BEC-ed atoms is given by formula (8) of [11]

$$|\psi_g\rangle = \frac{1}{\sqrt{N!}} \left( \sum_{m=-f}^f \xi_m a_{0,m}^\dagger \right)^N |vac\rangle \quad (32)$$

In the mean-field approximation the operators  $a_{0,m}^\dagger$ ,  $a_{0,m}$  are replaced by the  $c$ -numbers  $\xi_m \sqrt{N_0}$ , where  $N_0$  is the number of the condensed atoms. After this substitution, the initial Hamiltonian loses its global gauge symmetry and does not commute any longer with the operator  $\hat{N}$ . The order parameters  $\xi_m$  are chosen so as to minimize the expectation value of the new Hamiltonian as well as of its ground state and satisfy the normalization condition  $\sum_m |\xi_m|^2 = 1$ . To

keep the order parameter of each phase unchanged, it is necessary to specify the combination of the gauge transformation and spin rotations [11]. This program was carried out in [18-21].

The initial Hamiltonian (29) in the absence of the external magnetic field has the symmetry  $U(1) \times SO(3)$  representing the global gauge symmetry  $U(1)$  and the spin-rotation symmetry  $SO(3)$ . The generators of these symmetries are referred to as symmetry generators and have the form

$$\hat{N} = \int d\vec{x} \hat{\psi}_m^\dagger(x) \hat{\psi}_m(x) = \sum_{\vec{k}, m} a_{\vec{k}, m}^\dagger a_{\vec{k}, m} \quad (33)$$

$$\hat{F}^j = \int d\vec{x} \hat{\psi}_m(x) f_{mn}^j \hat{\psi}_n(x); \quad j = x, y, z$$

Unlike the  $SO(3)$  symmetry group with three generators  $\hat{F}^x$ ,  $\hat{F}^y$  and  $\hat{F}^z$ , the  $SO(2)$  symmetry group has only one generator  $\hat{F}^z$  which describes the spin rotation around the  $z$  axis and looks as follows:

$$\hat{F}^z = \sum_{\vec{k}, m} m a_{\vec{k}, m}^\dagger a_{\vec{k}, m} \quad (34)$$

In the presence of an external magnetic field, the symmetry of the Hamiltonian is  $U(1) \times SO(2)$ . The breaking of the continuous symmetries means the breaking of their generators. The number of the broken generators (BG) is denoted as  $N_{BG}$ . It equals 4 in the case of  $U(1) \times SO(3)$  symmetry and to 2 in the case of  $U(1) \times SO(2)$  symmetry.

The phase transition of the spinor Bose gas from the normal state to the Bose-Einstein condensed state was introduced mathematically into Hamiltonian (29) using the Bogoliubov displacement canonical transformation, when the single-particle creation and annihilation operators with a given wave vector  $\vec{k}$ , for example  $\vec{k} = 0$ , were substituted by the macroscopically  $c$ -numbers describing the condensate formation. The different superpositions of the single-particle states determine the structure of the finally established spinor phases [11]. Nielsen and Chadha [17] formulated a theorem which establishes the relation between the number of the Nambu-Goldstone modes, which must be present between the amount of the collective elementary excitations, which appear over the ground state of the system, if it is formed as a result of the spontaneous breaking of the  $N_{BG}$  continuous symmetries. The number of NG modes of the first type with linear (odd) dispersion law in the limit of long wavelengths denoted as  $N_l$  being accounted once, and the number  $N_{II}$  of the NG modes of the second type with quadratic (even) dispersion law at small wave vectors, being accounted twice give rise to the expression  $N_l + 2N_{II}$ , which is equal to or greater than the number  $N_{BG}$  of the broken symmetry

generators. The theorem is [17]

$$N_I + 2N_{II} \geq N_{BG} \quad (35)$$

It was verified in [11] for multiple examples of the spin  $-1$  and spin  $-2$  Bose-Einstein condensate phases. In the case of spin  $-2$  nematic phases, the special Bogoliubov modes that have linear dispersion relation but do not belong to the NG modes were revealed. The Bogoliubov theory of the spin  $-1$  and spin  $-2$  Bose-Einstein condensates (BECs) in the presence of the quadratic Zeeman effect was developed by Uchino, Kobayashi and Ueda [11] taking into account the Lee, Huang, Yang (LHY) corrections to the ground state energy, pressure, sound velocity and quantum depletion of the condensate. Many phases that can be realized experimentally were discussed to examine their stability against the quantum fluctuations and the quadratic Zeeman effect. The relations between the numbers of the NG modes and of the broken symmetry generators were verified. A brief review of the results concerning the spin  $-1$  phases of [11] is presented below so as to demonstrate, using these examples, the relations between the Bogoliubov excitations and the Nambu-Goldstone modes.

The first example is the ferromagnetic phase with  $c_1 < 0$ ,  $q < 0$  and the vector order parameter

$$\vec{\xi}^F = (1, 0, 0) \quad (36)$$

The modes with  $m=0$  and  $m=-1$  are already diagonalized, whereas the mode  $m=1$  is diagonalized by the standard Bogoliubov transformation. The Bogoliubov spectrum is given by formulas (33) and (34) of [11] repeated below

$$E_{\vec{k},1} = \sqrt{\varepsilon_{\vec{k}}(\varepsilon_{\vec{k}} + 2\eta(c_0 + c_1))}; \quad E_{\vec{k},0} = \varepsilon_{\vec{k}} - q; \quad E_{\vec{k},-1} = \varepsilon_{\vec{k}} - 2c_1 n \quad (37)$$

The  $E_{\vec{k},1}$  mode is massless. In the absence of a magnetic field when  $q=0$ , the mode  $m=0$  is also massless with the quadratic dispersion law. The initial symmetry of the Hamiltonian before the phase transition is  $U(1) \times SO(3)$ , whereas the final, remaining symmetry after the process of BEC is the symmetry of the ferromagnetic i.e.  $SO(2)$ . From the four initial symmetry generators  $\hat{N}$ ,  $\hat{F}^x$ ,  $\hat{F}^y$  and  $\hat{F}^z$  remains only the generator  $\hat{F}^z$  of the  $SO(2)$  symmetry. The generators  $\hat{F}^x$  and  $\hat{F}^y$  were broken by the ferromagnet phase, whereas the gauge symmetry operator  $\hat{N}$  was broken by the Bogoliubov displacement transformation. The number of the broken generators  $\hat{N}$ ,  $\hat{F}^x$ ,  $\hat{F}^y$  is three, i.e.,  $N_{BG}=3$ . In this case  $N_I=1$ ,  $N_{II}=1$  and  $N_I + 2N_{II} = 3$ , being equal to  $N_{BG}=3$ . The equality  $N_I + 2N_{II} = N_{BG}$  takes place. In the presence of an external magnetic field, with  $q \neq 0$ , the initial symmetry before the phase transition is  $U(1) \times SO(2)$  with two generators  $\hat{N}$  and  $\hat{F}^z$ , whereas after the BEC and the ferromagnetic phase formation the remained symmetry is  $SO(2)$ . Only one symmetry generator  $\hat{N}$  was broken. It means  $N_{BG}=1$ ,  $N_I=1$  and  $N_{II}=0$ . The equality  $N_I + 2N_{II} = N_{BG}$  also takes place.

For  $m=1$  Bogoliubov mode to be stable, the condition  $(c_0 + c_1) > 0$  is required. It ensures the mechanical stability of the meanfield ground state. Otherwise, the compressibility would not be positive definite and the system would become unstable against collapse. In the case  $q > 0$ ,  $c_1 > 0$  and  $(c_0 + c_1) < 0$  the state would undergo the Landau instability for the  $m=0$  and  $m=-1$  modes with quadratic spectra and the dynamical instability for the  $m=1$  mode with a linear spectrum (36)[11].

There are two polar phases. One with the parameters

$$\vec{\xi}^P = (0,1,0); \quad q > 0; \quad q + 2nc_1 > 0 \quad (38)$$

and the other with the parameters

$$\vec{\xi}^{P'} = \frac{1}{\sqrt{2}}(1,0,1); \quad q < 0; \quad c_1 > 0 \quad (39)$$

These two polar phases have two spinor configurations which are degenerate at  $q = 0$  and connect other by  $U(1) \times SO(3)$  transformation. However, for nonzero  $q$  the degeneracy is lifted and they should be considered as different phases. This is because the phase  $P$  has a remaining symmetry  $SO(2)$ , whereas the phase  $P'$  is not invariant under any continuous transformation. The number of NG modes is different in each phase and the low-energy behavior is also different. Following formulas (40)-(42) of [11] the density fluctuation operator  $a_{kd}$  and the spin fluctuation operators  $a_{k,f_x}$  and  $a_{k,f_y}$  were introduced

$$a_{kd} = a_{k,0}; \quad a_{k,f_x} = \frac{1}{\sqrt{2}}(a_{k,1} + a_{k,-1}); \quad a_{k,f_y} = \frac{i}{\sqrt{2}}(a_{k,1} - a_{k,-1}); \quad (40)$$

Their Bogoliubov energy spectra are

$$E_{\vec{k},d} = \sqrt{\varepsilon_{\vec{k}}(\varepsilon_{\vec{k}} + 2c_0)}; \quad E_{\vec{k},f_j} = \sqrt{(\varepsilon_{\vec{k}} + q)(\varepsilon_{\vec{k}} + q + 2nc_1)}; \quad (41)$$

In the presence of an external magnetic field, the initial symmetry is  $U(1) \times SO(2)$ , whereas after the BEC and the formation of the phase  $P$  with  $q \neq 0$  the remaining symmetry is also  $SO(2)$ . Only the symmetry  $U(1)$  and its generator  $\hat{N}$  were broken during the phase transition. It means we have in this case  $N_{BG} = 1$ ,  $N_I = 1$  and  $N_{II} = 0$ . The equality  $N_I + 2N_{II} = N_{BG}$  holds. Density mode is massless because the  $U(1)$  gauge symmetry is spontaneously broken in the mean-field ground state, while the transverse magnetization modes  $f_x$  and  $f_y$  are massive for non zero  $q$ , since the rotational degeneracies about the x and y axes do not exist being lifted by the external magnetic field. In the limit of infinitesimal  $q \rightarrow 0$  nevertheless nonzero, the transverse magnetization modes  $f_x$  and  $f_y$  become massless. It occurs because before the BEC in the absence of an external magnetic field the symmetry of the spinor Bose gas is  $U(1) \times SO(3)$ , whereas after the phase transition it can be considered as a remaining symmetry  $SO(2)$ . The generators  $\hat{N}$ ,  $\hat{F}^x$ ,  $\hat{F}^y$  were broken, whereas the generator  $\hat{F}^z$  remained. In this case we have  $N_{BG} = 3$ ,  $N_I = 3$  and  $N_{II} = 0$  the equality looks as  $3 = 3$ .

In the polar phase  $P'$  with the parameters (39) the density and spin fluctuation operators were introduced by formulas (57)-(59) [11]

$$a_{kd} = \frac{1}{\sqrt{2}}(a_{k,1} + a_{k,-1}); \quad a_{k,f_x} = a_{k,0}; \quad a_{k,f_y} = \frac{i}{\sqrt{2}}(a_{k,1} - a_{k,-1}); \quad (42)$$

with the Bogoliubov energy spectra described by formulas (65)-(67) [11]:

$$E_{\vec{k},d} = \sqrt{\varepsilon_{\vec{k}}(\varepsilon_{\vec{k}} + 2nc_0)}; \quad E_{\vec{k},f_x} = \sqrt{(\varepsilon_{\vec{k}} - q)(\varepsilon_{\vec{k}} - q + 2nc_1)}; \quad E_{\vec{k},f_z} = \sqrt{\varepsilon_{\vec{k}}(\varepsilon_{\vec{k}} + 2nc_1)} \quad (43)$$

At  $q < 0$  in contrast to the case  $q > 0$  one of the spin fluctuation mode  $E_{\vec{k},f_z}$  becomes massless.

The initial symmetry of the system is  $U(1) \times SO(2)$ . It has the symmetry generators  $\hat{N}$  and  $\hat{F}^z$ . They are completely broken during the phase transition. After the phase transition and the  $P'$

phase formation there are not any symmetry generators. The number of the broken generator is 2 ( $N_{BG} = 2$ ), whereas the numbers  $N_I$  and  $N_{II}$  are 2 and 0, respectively. As in the previous cases, the equality occurs in the Nielsen and Chadha rule. For the Bogoliubov spectra to be real the condition  $q < 0$ ,  $c_0 > 0$  and  $c_1 > 0$  must be satisfied, otherwise, the state  $\xi^P$  will be dynamically unstable.

Side by side with the spinor-type three-dimensional (3D) atomic Bose-Einstein condensates in the optical traps, we will discuss also the case of the Bose-Einstein condensation of the two-dimensional (2D) magnetoexcitons in semiconductors [22-25]. The collective elementary excitations in these conditions were investigated in [26-31]

The starting Hamiltonian (10) in [30] has two continuous symmetries. One is the gauge global symmetry  $U(1)$  and another one is the rotational symmetry  $SO(2)$ . The resultant symmetry is  $U(1) \times SO(2)$ . The gauge symmetry is generated by the operator  $\hat{N}$  of the full particle number, when it commutes with the Hamiltonian. It means that the Hamiltonian is invariant under the unitary transformation  $\hat{U}(\varphi)$  as follows

$$\hat{U}(\varphi)\hat{H}\hat{U}^{-1}(\varphi) = \hat{H}; \quad \hat{U}(\varphi) = e^{i\hat{N}\varphi}; \quad [\hat{H}, \hat{N}] = 0 \quad (44)$$

The operator  $\hat{N}$  is referred to as symmetry generator. The rotational symmetry  $SO(2)$  is generated by the rotation operator  $\hat{C}_z(\varphi)$  which rotates the in-plane wave vectors  $\vec{Q}$  on the arbitrary angle  $\varphi$  around  $z$  axis, which is perpendicular to the layer plane and is parallel to the external magnetic field. Coefficients  $W_{\vec{Q}}$ ,  $U(\vec{Q})$  and  $V(\vec{Q})$  in formulas (6) and (9) of [30] depend on the square wave vector  $\vec{Q}$  which is invariant under the rotations  $\hat{C}_z(\varphi)$ . This fact determines the symmetry  $SO(2)$  of the Hamiltonian (10). The gauge symmetry of Hamiltonian (10) [30] after the phase transition to the Bose-Einstein condensation (BEC) state is broken as it follows from expression (16) of [30]. In terms of the Bogoliubov theory of quasiaverages, it contains a supplementary term proportional to  $\tilde{\eta}$ . The gauge symmetry is broken because this term does not commute with the operator  $\hat{N}$ . More so, this term is not invariant under the rotations  $\hat{C}_z(\varphi)$ , because the in-plane wave vector  $\vec{k}$  of the BEC is transformed into another wave vector rotated by the angle  $\varphi$  in comparison with the initial position. The second continuous symmetry is also broken. In such a way, the installation of the Bose-Einstein condensation state with arbitrary in-plane wave vector  $\vec{k}$  leads to the spontaneous breaking of the both continuous symmetries. We will discuss the more general case  $\vec{k} \neq 0$  considering the case  $\vec{k} = 0$  as a limit  $\vec{k} \rightarrow 0$  of the cases with small values  $k \ll 1$ . One can remember, that the supplementary terms in Hamiltonian (10) of [30] describing the influence of the ELLs are actual in the range of small values  $kl < 0.5$ . Above we established that the number of the broken generators (BGs) denoted as  $N_{BG}$  equals to two ( $N_{BG} = 2$ ). Now we will discuss the number of the Nambu-Goldstone modes in a given system. Following the Goldstone's theorem in the systems with spontaneously broken continuous symmetries over the new ground states, there are some branches of the collective elementary excitations with gapless dispersion laws in the range of long wave-lengths. They are referred to as Nambu-Goldstone modes and can be compared with the massless particles in the relativistic physics. The Goldstone's theorem can be demonstrated considering a system described by the

complex scalar Bose field  $\phi(x) = \rho(x)e^{i\theta(x)}$  with the potential energy  $V(\phi)$  which has a nonlinear dependence on  $\phi$  in the form of a Mexican hat with the maximum at a point  $\phi = 0$  and with the minimal values forming a ring with the radius of the hat  $a$ . The minimal potential energy values in the classical description determine the energies of the ground states or, in other words, the vacua of the system. They are strongly degenerated in dependence on the phase  $\theta$  of the full operator. But under the influence of the quantum fluctuations between the manifold of vacua, one specific vacuum can be selected, for example, with the phase  $\theta = 0$ . In this case, the excitations of the system over the new vacuum state  $\theta = 0$  transiting it to an adjacent vacuum state with infinitesimal  $\theta \neq 0$  but lying on the vacuum ring will not need a finite amount of energy in the long wavelengths limit. The branch of the elementary excitations related only with the changes of the phases  $\theta$ , but with unchanged value of radius  $a$ , is gapless in the range of small wave vectors. Another branch of excitations related with the changes of the radius  $a$  of the ring need finite amount of energy and are gapped. They are compared with the massive particles in relativistic physics. The Nambu-Goldstone modes are classified as being of the first type (I) when their dispersion law is linear (odd) in dependence on the wave vector and of the second type (II) when this dependence is quadratic (even). Nielsen and Chadha [17] formulated a theorem, which establishes the relation between the numbers  $N_I$  and  $N_{II}$  of both types Nambu-Goldstone(NG) modes and the number of the broken symmetry generators  $N_{BG}$ . It affirms that the number of first type NG modes  $N_I$  being accounted once and the number of second type NG modes  $N_{II}$  being accounted twice is equal or prevails the number of broken generators  $N_{BG}$ . The theorem looks as follows:

$$N_I + 2N_{II} \geq N_{BG} \quad (45)$$

As was shown above, the spontaneous symmetry breaking yields Nambu-Goldstone modes, which play a crucial role in determining low-energy behavior of various systems [5, 32-37]. The Goldstone theorem guarantees that the NG modes do not acquire mass at any order of quantum corrections. Nevertheless, sometimes soft modes appear, which are massless in the zeroth order, but become massive due to quantum corrections. They were introduced by Weinberg [5], who showed that these modes emerge if the symmetry of an effective potential of the zeroth order is higher than that of the gauge symmetry, and the idea was invoked to account for the emergence of low-mass particles in relativistic physics. Following [32] now these modes are referred to as quasi-Nambu-Goldstone modes, in spite of the fact that their initial name introduced by Weinberg was pseudo-modes instead of quasi-modes. Georgi and Pais [33] demonstrated that the quasi-NG modes also occur in cases in which the symmetry of the ground state is higher than that of the Hamiltonian [32]. This type of the quasi-Nambu-Goldstone modes is believed to appear, for example, in the weak-coupled limit of A phase of  $^3\text{He}$  [37, 38].

#### 4. The Quasi-Nambu-Goldstone Modes in the Bose-Einstein Condensates

The authors of [32] underlined that the spinor BEC are ideal systems to study the physics of the quasi-NG modes, because these systems have a great experimental manipulability and well established microscopic Hamiltonian.

It was shown in [32] that the quasi-NG modes appear in a spin-2 nematic phase. In the nematic condensate, three phases, each of which has a different symmetry, are energetically

degenerate to the zeroth order [36] and the zeroth order solution has a rotational symmetry  $SO(5)$ , whereas the Hamiltonian of the spin-2 condensate has a rotational symmetry  $SO(3)$ . By applying the Bogoliubov theory of the BEC under the assumption that the  $\vec{k} = 0$  components of the field operators are macroscopically occupied, it was shown that the order parameter of the nematic phase has an additional parameter independent on the rotational symmetry.

The ground state symmetry of the nematic phase to a zeroth order approximation is broken by quantum corrections, thereby making the quasi-NG modes massive. The breaking of the  $SO(5)$  symmetry occurs. The number  $n$  of the quasi-NG modes was determined by Georgi and Pais [33] in the form of a theorem. It was explained and represented in [32] as follows:

$$n = \dim(\tilde{M}) - \dim(M) \quad (46)$$

where  $\tilde{M}$  is the surface on which the effective potential assumes its minimal values to the zeroth order and  $\dim(\tilde{M})$  is the dimension of this surface. The dimension  $\dim(M)$  determines the number of the NG modes. This implies that  $M$  is a submanifold of  $\tilde{M}$  and  $n$  is the dimension of the complementary space of  $M$  inside  $\tilde{M}$  [32].

In the case considered by Goldstone, the dimension of the ring is 1 and the number of the NG modes is 1; this leads to the absence of the quasi-NG modes ( $n = 0$ ). Returning to the case of 2D magnetoexcitons in the BEC state with wave vector  $\vec{k}$  different from zero ( $\vec{k} \neq 0$ ) described by Hamiltonian (16) of [30], one can remember that the both continuous symmetries existing in the initial form (10) [30] were lost. It happened due the presence of the term  $\tilde{\eta}(d_k^\dagger + d_k)$  in the frame of the Bogoliubov theory of quasiaverages. Nevertheless, the energy of the ground state as well as the self-energy parts  $\Sigma_{ij}(P, \omega)$ , which determine the energy spectrum of the collective elementary excitations depend only on the modulus of the wave vector  $\vec{k}$  and do not depend at all on its direction. All these expressions have a rotational symmetry  $SO(2)$ , in spite of the fact that Hamiltonian (16) of [30] has lost it. In our mind we have the condition described by Georgi and Pais [33] favoring the emergence of the quasi-NG modes. We are explaining the existence of the gapped, massive exciton-type branches of the collective elementary excitations obtained in our calculations just by these considerations.

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